A Online Appendix

A.1 An Example of a non-maximally safe acceptability correspondence that still satisfies all properties of comonotonicity

To see that we may have a non-maximally safe acceptability correspondence that still satisfies all properties of comonotonicity, consider the following example:

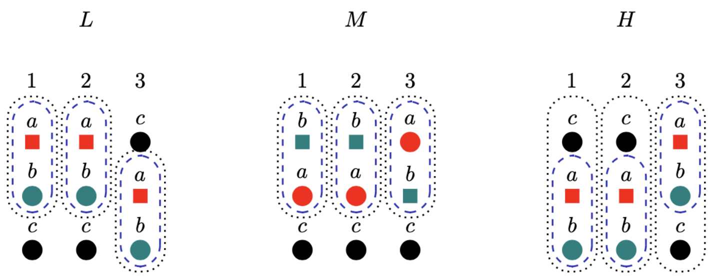

Figure 4: 1, 2 and 3's preference orderings over the three alternatives, at the three states, L, M, and H. For each state, the allocation chosen by SCF f^* in Ex. 6 is indicated by a square. The acceptability correspondence A from this example is shown by the dotted lines, and satisfies the conditions of Weak Comonotonicity. Acceptability correspondence A^* such that $A^*(\theta) = \{a, b\} \subseteq A(\theta)$ is maximally safe, and is represented by the dashed lines in the figure.

Example 6 Let everything be the same as the leading example, except at state L player 3's preference ordering is $c \succ a \succ b$. Let the SCF be $f^*(L) = f^*(H) = a$, $f^*(M) = b$, with $A(L) = A(M) = \{a, b\}$, and $A(H) = \{a, b, c\}$, as in the leading example. First note that, while it can be shown that f^* can be Safely implemented with respect to A, this acceptability correspondence is not *maximally safe*, since f^* can also be safely implemented with respect to the subcorrespondence A^*, such that $A^*(\theta) = \{a, b\}$ for all θ. Figure A.1 summarizes as usual agents' preferences, the SCC, and the two acceptability correspondences. Nonetheless, we show that (A, f^*), in this case, satisfies both conditions for comonotonicity. Part 1 of Def. 5 can be checked following the same logic as in the earlier examples (and it also follows from Proposition 1). To see that part 2 of Def. 5 also holds, note that it cannot be violated due to moving from states L or M to any other state, as $A(L) = A(M) \subseteq A(H)$, and therefore the condition is satisfied regardless. To see there is no violation moving from state H to state L, notice that relative to $f^*(H) = a$, an acceptable allocation at H, c, moves up in the ranking of player 3 from state H to L. Therefore we conclude that $A(H) \nsubseteq A(L)$ does not violate part 2 of Def. 5. To see that there is no violation moving from state H to state M, notice that relative to $f^*(H) = a$, an acceptable allocation at H, namely b, has moved up in player 1's ranking from state H to state M. With this, we also conclude that we do not violate part 2 of Def. 5 by setting

$A(H) \not\subseteq A(L)$. Hence, A is not maximally safe, and yet it is comonotonic with respect to f^*. $\square$

A.2 Proofs from Section 5.1 and 5.2

Proof of Result 1: Suppose for some $m^{\star,\theta}$ we have that $x \in g(B_{k-1}(m^{\star,\theta})) \cap \text{argmax}_{y \in A(\theta)} u_i(y, \theta')$ $\forall i \in N$. As $x \in g(B_{k-1}(m^{\star,\theta}))$ it follows that $\exists D_{k-1} \subset N_{k-1}, m_{D_{k-1}} \in M_{D_k}$ with $g(m_{D_{k-1}}, m^{\star,\theta}_{-D_{k-1}}) = x$.

Any unilateral deviation leads to an allocation in $A(\theta)$ by definition of (A, k)-Safe implementation and less than k agents are reporting a non-Equilibrium message. Therefore $m_{D_{k-1}}, m^{\star,\theta}_{-D_{k-1}}$ is a Nash Equilibrium at θ' and therefore $g(m_{D_{k-1}}, m^{\star,\theta}_{-D_{k-1}}) \in F(\theta')$. $\blacksquare$

Proof of Result 2: Let each agent $i \in N$ announce an outcome that is acceptable at some state, a state, and a natural number. Thus $M_i = \bigcup_{\theta'' \in \Theta} A(\theta'') \times \Theta \times \mathbb{N}$, with a typical element $m_i = (x^i, \theta^i, n^i)$. Let $g(m)$ be as follows:

(i) If $m_i = (x, \theta, n^i)$ $\forall i \in N$ and $x \in F(\theta)$ then $g(m) = x$

(ii) If $m_i = (x, \theta, n^i)$ $\forall i \in N \backslash \{j\}$ with $x \in F(\theta)$ and $m_j = (y, \cdot, \cdot)$ then

$$g(m) = \begin{cases} y & \text{if } y \in L_j(x, \theta) \cap A(\theta) \\ x & \text{if } y \notin L_j(x, \theta) \cap A(\theta) \end{cases}$$

(iii) $m_i = (x, \theta, \cdot)$, $x \in F(\theta)$, $\forall i \in N \backslash D$, $2 \leq |D| < \frac{n}{2}$ such that $\forall j \in D$ $m_j \neq (x, \theta, \cdot)$

$$g(m) = \begin{cases} x^{i^*} & \text{if } D^*(\theta, D) \neq \emptyset \\ x & \text{if } D^*(\theta, D) = \emptyset \end{cases}$$

where

$$D^*(\theta, D) = \{j \in D | x^j \in A(\theta)\}$$

and $i^* = \min\{i \in D^*(\theta, D) | n^i \geq n^j \quad j \in D^*(\theta, D)\}$

(iv) Otherwise, let $g(m) = x^{i^*}$ where $i^* = \min\{i \in N | n^i \geq n^j \quad \forall j \in N\}$

From here we can complete the proof in three steps: showing that all $x \in F(\theta)$ are induced by a Nash Equilibrium at θ, showing that there is no $y \notin F(\theta)$ such that y is induced by an Equilibrium at θ, and finally showing that the mechanism is indeed (A, k)-Safe.

Step 1. First to show that all $x \in F(\theta)$ are induced by Nash Equilibria at θ. Consider m^* such that $m^*_i = (x, \theta, \cdot)$, $\forall i \in N$ where $x \in F(\theta)$ at the state θ. To be a Nash Equilibrium we need to rule out the possibility that $\exists j \in N, m'_j \in M_j$ such that $u_j(g(m^*_{-j}, m'_j), \theta) > u_j(g(m^*), \theta)$.

However, $g(m^*_{-j}, m'_j) = y$ must be such that $y \in L_j(x, \theta)$ by rule (ii), it is not possible that $u_j(y, \theta) > u_j(x, \theta)$. Therefore it must be that m^* is a Nash Equilibrium leading to $x \in F(\theta)$.

Step 2. We will now show that no m^* a Nash equilibrium at θ that is a such that $g(m^*) = y \notin F(\theta)$. We proceed by showing that in each section of the rule, no Nash equilibrium leads to $y \notin F(\theta)$.

Case 1. Suppose m^* is a Nash equilibrium in rule i) at state θ such that $g(m^*) = y \notin F(\theta)$. It must be that $m^*_i = (y, \theta', n^i)$ for all $i \in N$ and, necessarily as $y \notin F(\theta)$, that $\theta' \neq \theta$. Given this, it must be that there is no profitable deviation and therefore, as deviations may only lead to rule (ii), it must be that for all $i \in N$, for any $z \in L_i(y, \theta') \cap A(\theta')$ we have that $z \in L_i(y, \theta)$, as there is no profitable deviation to report $m_i = (z, \theta, \cdot)$ inducing outcome z from rule (ii). With this, $L_i(y, \theta') \cap A(\theta') \subseteq L_i(y, \theta) \cap A(\theta')$. Therefore, by strong comonotonicity, we have that $y \in F(\theta)$, a contradiction.

Case 2. Now suppose that there is a Nash equilibrium m^*, which is in rule (ii), at state θ such that $g(m^*) = y \notin F(\theta)$. It must be that $\exists j \in N$ such that, $\forall i \in N \backslash \{j\}$ we have $m^*_i = (x, \theta', n^i)$, while $m^*_j \neq (x, \theta', \cdot)$. For this to be a Nash equilibrium it must be that there is no incentive for any agent to deviate. If $k > 1$ a deviation can lead to rule (i), (ii), or (iii), regardless, as m^* is a Nash equilibrium at θ, no agent $i \neq j$ to wish to change their report, inducing rule (iii), it must be that $y \in \operatorname{argmax}_{z \in A(\theta')} u_i(z, \theta)$. By weak Safe No-Veto, it must therefore be that $y \in F(\theta)$, a contradiction to $y \notin F(\theta)$. For $k = 1$ we have that a deviation can lead to rule (i), (ii), or (iv), which in the case of rule (iv) can induce any outcome. Those that can deviate to impose rule (iv) are all agents other than j. With this, we have that, as there is no incentive to deviate, that $y \in \operatorname{argmax}_{z \in \bigcup_{\theta'' \in \Theta} A(\theta'')} u_i(z, \theta)$ for all $i \in N \backslash \{j\}$. With this, it must be that $y \in \operatorname{argmax}_{z \in A(\theta')} u_i(z, \theta)$ for all $i \in N \backslash \{j\}$, and therefore by weak Safe No-Veto we have that $y \in F(\theta)$, a contradiction.

Case 3. Notice that there can be no Nash equilibria within rule (iii). Suppose that m^* were a Nash equilibrium in rule (ii) at state θ. Suppose that $|D| < \lfloor \frac{n}{2} \rfloor$ and $m^*_i = (x, \theta' . \cdot)$ for all agents $i \notin D$. Given this, it must be that there is no profitable deviation for any agent. As there exists a message for any player that leads to any allocation in $A(\theta')$ via rule (iii), we conclude that $y \in \operatorname{argmax}_{z \in A(\theta')} u_i(z, \theta)$ for all $i \in N$. Therefore it must be that no unanimity in A is violated. Now suppose that $|D| = \lfloor \frac{n}{2} \rfloor$. For there to be no profitable deviation, it must be that for $\forall i \in D$, $y \in \operatorname{argmax}_{z \in A(\theta')} u_i(z, \theta)$. For all agents in $i \in N \backslash D$ it must be that for any $x \in X \supseteq A(\theta')$, we have that $u_i(y, \theta) \geq u_i(x, \theta)$, as there is no profitable deviation. Given this, we conclude that $y \in \operatorname{argmax}_{z \in A(\theta')} u_i(z, \theta)$ for all $i \in N$, and therefore no unanimity in A is violated.

Case 4. Note that there can be no Nash equilibria within rule (iv). To see this, suppose that m^* is a Nash equilibrium at state θ that falls within rule (iv), with $g(m^*) = y$. Notice that any agent can deviate to remain within rule (iv), inducing any outcome that is acceptable at any state. Therefore for y to be a Nash equilibrium it must be that $y \in \text{argmax}_{z \in \bigcup_{\theta'' \in \Theta} A(\theta'')} u_i(z, \theta)$ for all $i \in N$. Therefore it follows that $y \in \text{argmax}_{z \in A(\theta')} u_i(z, \theta)$ for some $\theta' \in \Theta$ and for all $i \in N$. Therefore no unanimity in A is violated.

Step 3.

We will now show that all Nash equilibria are safe. To do so, we will again split it into cases. By the previous analysis, recall that if we maintain No Unanimity in A we know that there can only be equilibria in rule (i) or rule (ii), and therefore we need only focus on the safety of those equilibria in rules (i) and (ii).

Case 1. If m^* is a Nash equilibrium at θ that falls into rule (i) it must be that $m_i^* = (y, \theta', n^i)$. By the previous analysis, we know that $y \in F(\theta)$. If $\theta' = \theta$, we conclude that safety is satisfied as k deviations can only lead to rule (ii) or (iii). Either way, we remain in $A(\theta)$. Now suppose that $\theta' \neq \theta$ while m^* is a Nash equilibrium at θ. Notice that regardless, k deviations must lead to remaining within $A(\theta')$ via rule (ii) or (iii). By the previous analysis, we know that this only occurs when $L_i(y, \theta') \cap A(\theta') \subseteq L_i(y, \theta) \cap A(\theta')$ for all $i \in N$. Given this, $A(\theta') \subseteq A(\theta)$ must hold for strong comonotonicity to be satisfied. Given that $A(\theta') \subseteq A(\theta)$, we conclude that any deviation from such a Nash equilibrium must remain in $A(\theta')$, and therefore $A(\theta)$, maintaining safety.

Case 2. Now suppose that m^* is a Nash equilibrium at θ that falls into rule (ii). It must be that $\forall i \neq j$ $m_i^* = (x, \theta', n^i)$ while $m_j^* \neq (x, \theta', n^i)$. Notice that k deviations can lead to rule (i), rule (ii) or rule (iii), as $k < \lfloor \frac{n}{2} \rfloor - 1$. By the structure of the mechanism, even with k, in the extreme case where $\frac{n}{2} - 2$ misreports from m^*, it remains that the majority of agents are reporting $m_i = (x, \theta', n^i)$. With this, any k deviations must lead to $A(\theta')$. Notice that for this to be a Nash equilibrium at θ, we therefore require that $g(m^*) = y \in \text{argmax}_{z \in A(\theta')} u_i(z, \theta)$ for all $i \neq j$. With this, by Weak Safe No-Veto, we have that $A(\theta') \subseteq A(\theta)$. As k deviations remain in $A(\theta')$ it is also true that k deviations remain in $A(\theta)$. Therefore Safety is upheld. ■

Proof of Result 3: Let each agent $i \in N$ announce an outcome, a state, and a natural number. Thus $M_i = X \times \Theta \times \mathbb{N}$, with a typical element $m_i = (x^i, \theta^i, n^i)$. Let $g(m)$ be as follows:

(i) If $m_i = (x, \theta, n^i)$ $\forall i \in N$ and $x \in F(\theta)$ then $g(m) = x$

(ii) If $m_i = (x, \theta, n^i)$ $\forall i \in N \backslash \{j\}$ with $x \in F(\theta)$ and $m_j = (y, \cdot, n^j)$ then

$$g(m) = \begin{cases} \frac{n^j}{n^j+1} y + \frac{1}{n^j+1} x & \text{if } y \in L_j(x, \theta) \cap A(\theta) \\ x & if y \notin L_j(x, \theta) \cap A(\theta) \end{cases}$$

(iii) if $m_i = (x, \theta, \cdot)$, $x \in F(\theta)$, $\forall i \in N \backslash D$, $2 \leq |D| \leq \frac{n}{2}$ such that $\forall j \in D$ $m_j \neq (x, \theta, \cdot)$

$$g(m) = \begin{cases} \sum_{y \in A(\theta)} \frac{1}{|A(\theta)| + \sum_{k \in D^*(\theta, D)} n^k} y + ... \\ +... \sum_{j \in D^*(\theta, D)} \frac{n^j}{|A(\theta)| + \sum_{k \in D^*(\theta, D)} n^k} x^j & \text{if } D^*(\theta, D) \neq \emptyset \\ \sum_{y \in A(\theta)} \frac{1}{|A(\theta)|} y & \text{if } D^*(\theta, D) = \emptyset \end{cases}$$

where $D^*(\theta, D) = \{j \in D | x^j \in A(\theta)\}$.

(iv) Otherwise, let $g(m) = \sum_{x \in X} \frac{1}{|X| + \sum_{j \in N} n^j} x + \sum_{i \in N} \frac{1}{|X| + \sum_{j \in N} n^j} x^i$.

From here we can complete the proof in three steps: showing that all $x \in F(\theta)$ are induced by a Nash Equilibrium at θ, showing that there is no $y \notin F(\theta)$ such that y is induced by an Equilibrium at θ, and finally showing that the mechanism is indeed (A, k)-Safe.

Step 1. First to show that all $x \in F(\theta)$ are induced by Nash Equilibria at θ. Consider m^* such that $m_i^* = (x, \theta, \cdot), \quad \forall i \in N$ where $x \in F(\theta)$ at the state θ. To be a Nash Equilibrium we need to rule out the possibility that $\exists j \in N, m_j' \in M_j$ such that $u_j(g(m_{-j}^*, m_j'), \theta) > u_j(g(m^*), \theta)$.

By rule (ii), the only way that $g(m_{-j}^*, m_j') \neq x$, i.e. not to give the deterministic allocation x, it must be that it puts positive weight on x and on one other allocation $y \in L_j(x, \theta) \cap A(\theta)$. Given that $y \in L_j(x, \theta)$, there is no profitable deviation.

Step 2. We will now show that no m^* a Nash equilibrium at θ that is such that $g(m^*) \notin F(\theta)$, i.e. no Nash equilibrium gives anything but the deterministic allocations of $F(\theta)$. We proceed by showing that in each section of the rule, no Nash equilibrium leads to any $y \notin F(\theta)$, or any probabilistic allocation.

Case 1. Suppose m^* is a Nash equilibrium in rule i) at state θ such that $g(m^*) = y \notin F(\theta)$. It must be that $m_i^* = (y, \theta', n^i)$ for all $i \in N$ and, necessarily as $y \notin F(\theta)$, that $\theta' \neq \theta$. Given this, it must be that there is no profitable deviation, and therefore, as deviations may only lead to rule (ii), it must be that for all $i \in N$, for any $z \in L_i(y, \theta') \cap A(\theta')$ we have that $z \in L_i(y, \theta)$, as there is no profitable deviation to report $m_i = (z, \theta, \cdot)$ inducing outcome z from rule (ii). With this, $L_i(y, \theta') \cap A(\theta') \subseteq L_i(y, \theta) \cap A(\theta')$. Therefore, by strong comonotonicity, we have that $y \in F(\theta)$, a contradiction.

Case 2. Now suppose that there is a Nash equilibrium m^*, which is in rule (ii), at state θ such that $g(m^*) \notin F(\theta)$. It must be that $\exists j \in N$ such that, $\forall i \in N \backslash \{j\}$ we have $m_i^* = (x, \theta', n^i)$, while $m_j^* \neq (x, \theta', \cdot)$. We split this possibility into sub-cases for clarity.

Case 2.a. First consider the case that $g(m^*) = x \in F(\theta')$. It must therefore be that for all $i \neq j$ there is no profitable deviation. Given this, we must have that $u_i(x, \theta) \geq \max_{z \in A(\theta') \backslash \{x\}} u_i(z, \theta)$ by the fact a deviation to announce an arbitrarily high n^i, therefore, inducing a probabilistic outcome putting almost probability 1 on their most preferred outcome z. Further, for j to have no profitable deviation we have that it must be that there is no $y \in L_j(x,\theta')\cap A(\theta')$ such that $u_j(y,\theta) > u_j(x,\theta)$. Therefore for all $y \in L_j(x,\theta')\cap A(\theta')$ we have that $u_j(x,\theta) \geq u_j(y,\theta)$, and therefore it is the case that $L_j(x,\theta') \cap A(\theta') \subseteq L_j(x,\theta) \cap A(\theta')$. Further, we have that $L_i(x,\theta) \cap A(\theta') = A(\theta')$ for all $i \neq j$. With this $L_i(x,\theta') \cap A(\theta') \subseteq L_i(x,\theta) \cap A(\theta')$ for all $i \in N$. Therefore by Strong Comonotonicity we have that $x \in F(\theta)$ and $A(\theta') \subseteq A(\theta)$.

Case 2.b. Now instead consider the case where $g(m^*) = \frac{n^j}{n^j+1}y + \frac{1}{n^j+1}x$. As for all $\theta, \theta' \in \Theta$, for all $z \in F(\theta')$ $z' \in A(\theta')$, $\exists i \in N$ such that $u_i(z,\theta) - u_i(z',\theta) \neq 0$, it must be that case that agent is such that $u_i(y,\theta) - u_i(x,\theta) \neq 0$. If such agent is j, i.e. the whistle blower, then a profitable deviation exists to announce either a higher n^j, putting more weight on y, or announce $m'_j = m^*_i$ for $i \neq j$, putting weight 1 on x. Now suppose that $u_i(y,\theta) - u_i(x,\theta) \neq 0$ for $i \neq j$, while $u_j(y,\theta) - u_i(x,\theta) = 0$. Firstly, suppose that $u_i(y,\theta) > u_i(x,\theta)$. Notice that $\forall \epsilon > 0$ $\exists n^i \in \mathbb{N}$ such that $\epsilon > \frac{|A(\theta')|}{|A(\theta')|+n^i+n^j}(u_i(y,\theta) - \min_{z\in A(\theta')} u_i(z,\theta))$. Therefore, simply rearranging this, we have that $\forall \epsilon > 0$ $\exists n^i \in \mathbb{N}$ such that $\frac{n^i+n^j}{|A(\theta')|+n^i+n^j}u_i(y,\theta) + \frac{|A(\theta')|}{|A(\theta')|+n^i+n^j}\min_{z\in A(\theta')} u_i(z,\theta) > u_i(y,\theta) - \epsilon$. Given this, we conclude that $\forall \epsilon > 0$ $\exists n^i \in \mathbb{N}$ such that $\frac{n^i+n^j}{|A(\theta')|+n^i+n^j}u_i(y,\theta) + \sum_{z\in A(\theta')} \frac{1}{|A(\theta')|+n^i+n^j}u_i(z,\theta) > u_i(y,\theta) - \epsilon$. Let $\epsilon = u_i(y,\theta) - \frac{n^j}{n^j+1}u_i(y,\theta) - \frac{1}{n^j+1}u_i(x,\theta)$. By assumption that $u_i(y,\theta) - u_i(x,\theta) > 0$ we have that $\epsilon > 0$. With this, $\exists n^i \in \mathbb{N}$ such that $\frac{n^i+n^j}{|A(\theta')|+n^i+n^j}u_i(y,\theta) + \sum_{z\in A(\theta')} \frac{1}{|A(\theta')|+n^i+n^j}u_i(z,\theta) > \frac{n^j}{n^j+1}u_i(y,\theta) + \frac{1}{n^j+1}u_i(x) = u_i(g(m^*),\theta)$. With this, announcing $m'_i = (y,\theta,n^i)$ induces such an outcome

$$u_i(g(m'_i, m^*_{-i}),\theta) = \frac{n^j}{n^j+1}u_i(y,\theta) - \frac{1}{n^j+1}u_i(x) > u_i(g(m^*),\theta)$$

and therefore m^* cannot be an equilibrium. By an analogous argument, there cannot be an equilibrium if $u_i(x,\theta) > u_i(y,\theta)$ for some agent, as they can announce an arbitrarily high n^i and x, putting almost probability 1 on x. Regardless, this m^* such that $g(m^*) = \frac{n^j}{n^j+1}y + \frac{1}{n^j+1}x$ cannot be an equilibrium.

Case 3 and 4. Note that there cannot be any equilibria in rule (iii) or rule (iv). To see this, notice that any agent can announce their most one of their most preferred outcome from $A(\theta)$, in rule (iii), or X in rule (iv), and an integer higher than any other agent (including themselves before the deviation), and strictly increase their utility by reducing the probability assigned to their less preferred option. As at least one agent is not completely indifferent between all allocations by No total indifference across F and A, one such agent always exists.

Step 3. Notice that by the previous analysis, there may on be equilibria in rules (i) and (ii), therefore we need only check the Safety of such equilibria.

Case 1. Firstly, suppose that m^* is a Nash equilibrium in rule (i) at state θ. By the previous analysis, we know that it is the case that $m^*_i = (x,\theta',\cdot)$, with $x \in F(\theta')$. If $\theta' \neq \theta$, then, by the previous analysis, we know that $L_i(x,\theta')\cap A(\theta') \subseteq L_i(x,\theta)\cap A(\theta')$ for all $i \in N$. Therefore by Strong Comonotonicity we have that $x \in F(\theta)$ and $A(\theta') \subseteq A(\theta)$. Now notice that in k deviations from m^*, we may only reach $A(\theta')$, via rule (ii), with 1 deviation, or rule (iii) which can be reached with more than 1 but less than $k+1$ deviations. As $k < \frac{n}{2} - 1$, it is the case that the majority of agents still report m_i^*, regardless of what the other k report. Given that $A(\theta') \subseteq A(\theta)$, it follows that any allocation with k deviations of m^* is still a mix with a support of $A(\theta)$. If instead m^* is such that $\theta' = \theta$, Safety is upheld as k deviations can only lead to stochastic allocations over $A(\theta)$. Therefore Safety is upheld.

Case 2. Now instead suppose m^* is a Nash equilibrium at state θ that falls into rule (ii). It must be that $m_i^* = (x, \theta', \cdot)$ for all $i \neq j$ and $m_j^* = (y, \theta'', \cdot)$. By the previous analysis, we know that $g(m^*) = x$. If $\theta' \neq \theta$, again by the previous analysis, we know that it must be that $L_i(x, \theta') \cap A(\theta') \subseteq L_i(x, \theta) \cap A(\theta')$ for all $i \in N$. Therefore by strong comonotonicity we have that $x \in F(\theta)$ and $A(\theta') \subseteq A(\theta)$. Now notice that in k deviations we may reach rule (i) inducing x, rule (ii) inducing mixes over allocations in $L_j(x, \theta') \cap A(\theta')$, or rule (iii) for inducing stochastic allocations over $A(\theta')$. Notice that no other allocations can be reached in k deviations as $k < \frac{n}{2} - 1$, and therefore the majority of agents would still be reporting m_i^*. With this, and by $A(\theta') \subseteq A(\theta)$, we have that the mechanism is still considered Safe. Similarly, if $\theta' = \theta$, we have that in k deviations we may reach rule (i) inducing x, rule (ii) inducing mixes over allocations in $L_j(x, \theta) \cap A(\theta)$, or rule (iii) for inducing stochastic allocations over $A(\theta)$. With this, Safety is upheld.■

Proof of Result 4: Take the mechanism and logic to be similar to that of theorem 3:

Let each agent $i \in N$ announce an outcome that is acceptable at some state, a state, and a natural number. Thus $M_i = \bigcup_{\theta'' \in \Theta} A(\theta'') \times \Theta \times \mathbb{N}$, with a typical element $m_i = (x^i, \theta^i, n^i)$. Let $g(m)$ be as follows:

(i) If $m_i = (x, \theta, n^i)$ $\forall i \in N$ and $x \in F(\theta)$ then $g(m) = x$

(ii) If $m_i = (x, \theta, n^i)$ $\forall i \in N \backslash \{j\}$ with $x \in F(\theta)$ and $m_j = (y, \theta', \cdot)$ then

$$g(m) = \begin{cases} y & \text{if } u_i(x, \theta, (x, \theta, \cdot)) \geq u_i(y, \theta, (y, \theta', \cdot)) \text{ and } y \in A(\theta) \\ x & \text{if either } u_i(x, \theta, (x, \theta, \cdot)) < u_i(y, \theta, (y, \theta', \cdot)) \text{ or } y \notin A(\theta) \end{cases}$$

(iii) If $m_i = (x, \theta, \cdot)$, $x \in F(\theta)$, $\forall i \in N \backslash D$, $2 \leq |D| < \frac{n}{2}$ such that $\forall j \in D$ $m_j \neq (x, \theta, \cdot)$

$$g(m) = \begin{cases} x^{i^*} & \text{if } D^*(\theta, D) \neq \emptyset \\ x & \text{if } D^*(\theta, D) = \emptyset \end{cases}$$

where

$$D^*(\theta, D) = \{j \in D | x^j \in A(\theta)\}$$

and $i^* = \min\{i \in D^*(\theta, D) | n^i \geq n^j \quad j \in D^*(\theta, D)\}$

(iv) Otherwise, let $g(m) = x^{i^*}$ where $i^* = \min\{i \in N | n^i \geq n^j \quad \forall j \in N\}$

From here we can complete the proof in three steps: showing that all $x \in F(\theta)$ are induced by a Nash Equilibrium at θ, showing that there is no $y \notin F(\theta)$ such that y is induced by an Equilibrium at θ, and finally showing that the mechanism is indeed (A, k)-Safe. We will proceed by showing that all Nash Equilibria are contained in rule (i), and report the correct state, and therefore, in comparison to theorem 3, we may weaken Safe No-Veto to only Unanimity within all acceptable allocations.

Step 1. First to show that all $x \in F(\theta)$ are induced by Nash Equilibria at θ. Consider m^* such that $m_i^* = (x, \theta, \cdot), \quad \forall i \in N$ where $x \in F(\theta)$ at the state θ. To see this is a NE, suppose not there is some agent for which there is a profitable deviation m_j', $g(m_{-j}^*, m_j') = y$ must be such that $u_i(x, \theta, (x, \theta, n)) \geq u_i(y, \theta, (y, \theta', n))$ and $y \in A(\theta)$ (or it is not profitable) by rule (ii), a contradiction to $u_j(y, \theta, (y, \theta', n)) > u_j(x, \theta, (x, \theta, n))$. Therefore it must be that m^* is a Nash Equilibrium leading to $x \in F(\theta)$.

Step 2. We will now show that no m^* a Nash equilibrium at θ that is a such that $g(m^*) = y \notin F(\theta)$. We proceed by showing that in each section of the rule, no Nash equilibrium leads to $y \notin F(\theta)$.

Suppose m^* is a Nash equilibrium in rule i) at state θ such that $g(m^*) = y \notin F(\theta)$. It must be that $m_i^* = (y, \theta', n^i)$ for all $i \in N$ and, necessarily as $y \notin F(\theta)$, that $\theta' \neq \theta$. However, consider a deviation for player i to a report of $m_i = (y, \theta', \cdot)$. This induces the outcome y still. By the definition of weak preference for correctness, we have that $u_i(y, \theta', (y, \theta', \cdot)) > u_i(y, \theta', (y, \theta, \cdot))$, and therefore a profitable deviation exists. A contradiction that m^* being an equilibrium.

Suppose that we have an Equilibrium in case (ii) with $m_i^* = (x, \theta, n^i)$ for all $i \neq j$ and $m_j^* = (y, \cdot, \cdot)$. Suppose the true state is θ'. For this to be the case, it must be that no agent has an incentive to deviate. Therefore it must be that $g(m^*) \neq x$, as otherwise j has an incentive to deviate by announcing $m_j = (x, \theta, n^j)$, and by a preference for correctness would now be announcing the correct state and / or allocation that the mechanism implements. Therefore it must be that $g(m^*) = y$. However, given this, any agent $i \neq j$ has the incentive to deviate to $m_i = (y, \theta', n^i)$, and therefore leading to allocation y via rule (iii) or rule (iv). Via the weak preference for correctness, this strictly increases utility. Therefore there can be no equilibria in rule (ii).

Suppose that the Equilibrium m^* at state θ' is in rule (iii), with $m_i^* = (x, \theta, \cdot)$, $x \in F(\theta)$, $\forall i \in N \backslash D$, $2 \leq |D| < \frac{n}{2}$ such that $\forall j \in D$ $m_j^* \neq (x, \theta, \cdot)$. It must be that at least $|D|$ agents are such that they are either reporting the wrong state or not reporting the allocation that is being implemented $g(m^*) = y$, be that those in D or those in $N \backslash D$. Given this, consider one such agent. They may report the allocation y and/or the state θ' and an integer higher than any other agent. To see this does not change the allocation first consider the case $\frac{n}{2} > |D| > 2$. In such a case, we remain in rule (iii) or rule (iv) via this deviation, where the deviating agent is announcing the highest integer and therefore y is allocated. Now consider the case where $|D| = 2$. First consider $g(m^*) = y = x$. Suppose that $\theta \neq \theta'$, then any agent in $N \backslash D$ may deviate to announce $m_i = (x, \theta', n^i)$ with n^i being higher than any integer announced under m^*. As this announcement announces the true state, it strictly increases the utility of i. Therefore it cannot be that m^* is a Nash equilibrium in this case. Suppose instead that $\theta = \theta'$. Then it must be that those in D are either:

1. Both announcing an allocation not in $A(\theta)$, in which case a deviation by either to $m_j = (x, \theta, \cdot)$ would not change the allocation but would make the report correct, therefore increasing their utility.

2. One is announcing an allocation in $A(\theta)$, while one is not. In which case, there is at least one who is not announcing x, in which case they can increase their utility by doing so.

3. Both are announcing allocations in $A(\theta)$. If this is the case, if both announce x it must be that both are announcing $\theta^j \neq \theta$, and therefore can improve their utility by announcing $m_j = (x, \theta, \cdot)$, and increase their utility, leading to rule (ii), but keeping the same allocation. Now suppose that only one is announcing x. It must be that the other is not, and therefore can increase their utility by announcing x, while keeping the other parts of the report the same, strictly increasing their utility. If neither is announcing x, it cannot be that $g(m^*) = x$.

Now instead consider $g(m^*) = y \neq x$. In such a case, those outside of D may deviate to announce $m_i = (y, \theta', \cdot)$, increasing their utility.

Finally, consider the possibility of an equilibrium m^* in rule (iv) at state θ leading to the outcome y. For this to be the case, it must be that there is no incentive to deviate. Consider the possibility that $m_i^* \neq (y, \theta, \cdot)$ for some i. For this to be the case, it must be that announcing $(y, \theta, \cdot)$ and an integer higher than any other announced under m^* would change the allocation, as otherwise, the preference for correctness would mean a profitable deviation occurs. This can only occur if such a deviation would lead to rule (iii), i.e. $\lfloor \frac{n}{2} \rfloor - 1$ agents are reporting $(x, \theta', \cdot)$. Given this, we can deduce that $(x, \theta', \cdot) = (y, \theta, \cdot)$ and that $y \in F(\theta)$, as otherwise, rule (iv) would dictate the allocation remains the same, while at least one of those $\lfloor \frac{n}{2} \rfloor - 1$ agents could strictly increase their utility by announcing $m_j = (y, \theta, \cdot)$. With this, the original player i such that $m_i^* \neq (y, \theta, \cdot)$ has a profitable deviation as they can announce $(y, \theta, \cdot)$ and some arbitrarily high n^i, inducing rule (iii), where y is chosen due to i announcing the highest integer and $y \in F(\theta) \subseteq A(\theta)$. Therefore it cannot be that $m_i^* \neq (y, \theta, \cdot)$ for any i in any equilibria in rule (iv). For such equilibria to fall into rule (iv) rather than rule (i), it must be that $y \notin F(\theta)$. However, for there to be no profitable deviation within this rule it must therefore be that $y \in \operatorname{argmax}_{x \in \bigcup_{\theta'' \in \Theta} A(\theta'')} u_i(x, \theta, m_i)$ for all $i \in N$, and therefore by Unanimity within all Acceptable Allocations we have that $y \in F(\theta)$. ■